GROWTH AND SIZE DISTRIBUTION OF FIRMS IN AN INDUSTRY

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SUMMARY

The growth and distribution of business firms by sizes within a particular industry has received considerable attention from economists and statisticians interested in phenomena such as competition, oligopoly, concentration measures, etc. The application of stochastic techniques to the explanation of these dynamic phenomena has been successful. This paper is devoted to the construction of some stochastic models to predict the size distribution of firms in a particular industry.

In the models discussed we assumed that the size distribution of firms in a given industry between an enumerable infinity of firm-size ranges is assumed to develop by means of a stochastic process. In most of the models the stochastic matrix was assumed to remain constant over time. Under these circumstances and provided certain other conditions are satisfied, the distribution will tend toward a unique equilibrium dependent upon the stochastic matrix.

It was found that under fairly general conditions, provided that the prospect of change of firm sizes as described by the stochastic matrix is in a certain way independent of the size of the firms themselves above some limit, the equilibrium distribution of firm sizes in a given industry is the Pareto distribution. Consideration was also given to the effects the levels of technological progress reached by the firms within an industry could have on the firm sizes equilibrium distribution.

Finally, the Pareto distribution is used to determine the equilibrium distribution of firm sizes in the computing accounting industry.

CHAPTER I

INTRODUCTION

The General Problem

The object of this paper is to apply stochastic processes in the construction and application of some models to the analysis of the structure which the biggest firms of a given industry would actually reach if certain trends were to continue. This may be accomplished assuming the distribution of firm-sizes through an enumerable infinity of firm-size ranges to be developed by means of a stochastic process. By assuming certain restrictions with respect to the rate of growth of firms we obtain a steady-state distribution of the size of firms in a given industry: the Pareto distribution. It is anticipated that the limitations of the models and the extent to which they approximate reality are to be ascertained during this study.

Purpose

The purpose of predicting the size distribution of firms in a given industry is to advance knowledge concerning the application of stochastic processes for the analysis of economic phenomena that cannot be advanced by the traditional theory of the firm.

A specific purpose is to characterize the large amount of factors which affect a market structure of an industry as a simple stochastic model. Another purpose is to suggest areas in which further analytical research may be taken up and to discuss implications of the stochastic processes when used to explain the behavior of the firms of a given industry.

Definition of the Problem

The motivation for studying the problem is essentially the following: the distribution of business firms by size is important in problems of competition and monopoly and also in issues of antitrust laws or concentration of industries. So far there does not exist a satisfactory index to measure concentration in industry.

The traditional theory of the firm that confines the firm within a static frame provided by categories of monopoly and competition, and by problems of output determination attempts to explain these dynamic phenomena about concentration and growth of firms in terms of static cost curves-long run average cost curve-with questionable results.

In recent years the use of stochastic processes in the explanation and formulation of models of growth in size distributions of firms has occupied a prominent place in the literature of size distribution of business firms, particularly because through the stochastic analysis it is possible to interpret the size distribution of firms in terms of the dynamics of the growth process rather than in terms of static cost curves. Also, if the assumptions on which the stochastic models rest are correct the models call for new statistical measures of the degree of concentration and new economic interpretations of concentrations. Bonini (21) recalls that if it is possible to fit a distribution function to the size of firms in a given industry on the basis of a theoretical model, it is reasonable to base the measures of concentration on the parameters of the distribution function.

CHAPTER II

LITERATURE SURVEY

It is not necessary to emphasize the difficulties of building models of growth and size distribution of firms. A firm is a complex organization and cannot only expand internally but also influence its environment to ensure growth possibilities. If all the factors that influence the growth of a firm were taken into account, the complexity of the model would make it unmanageable. On the other hand, if strong simplifications were made the model would lose reality.

To overcome these problems and given that the classical theory of the firm has little to say about the dynamic behavior of the firm, some efforts have been made in recent years toward the construction of models of growth and size distribution of firms, considering the growth of firms as a stochastic process.

Some of the most significant contributions are summarized in the next pages.

Stochastic Approaches

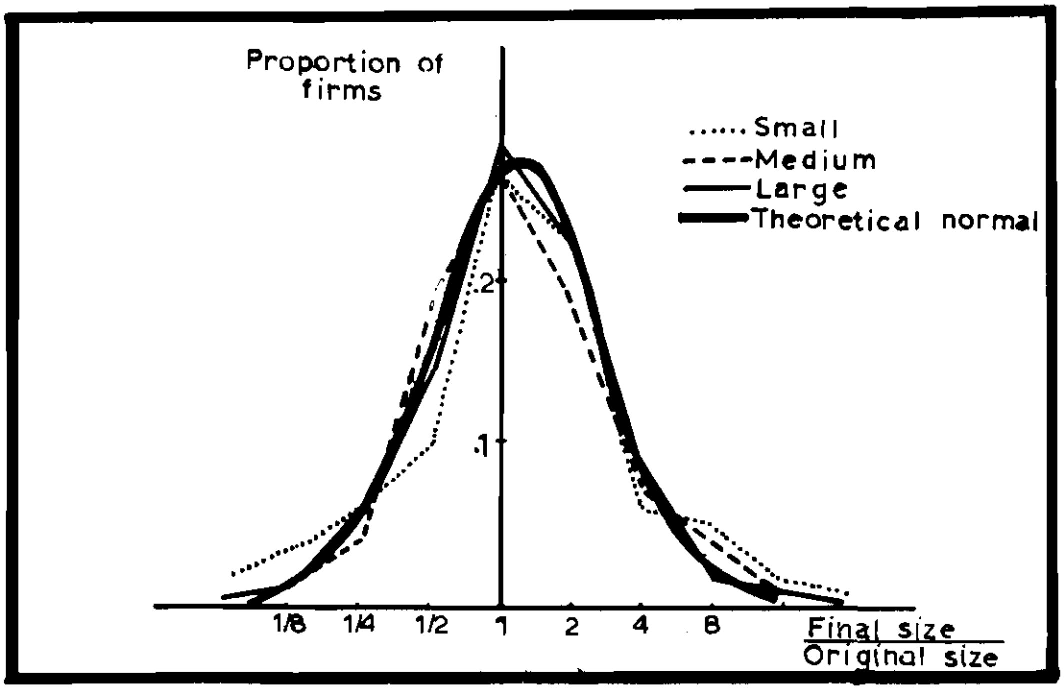
Hart and Prias (11), making studies of business concentration in the United Kingdom, discovered some interesting relationships between the size and growth of the British firms. Specifically, that the growth of the British firms which are alive both at the beginning and at the end of a given period of time can be represented by a single probability scheme described by the law of the proportionate growth,or Gibrat's law (3):

while a larger firm may have a better change of increasing its size by a given amount, the chance of a proportionate increase is the same for firms of all sizes.

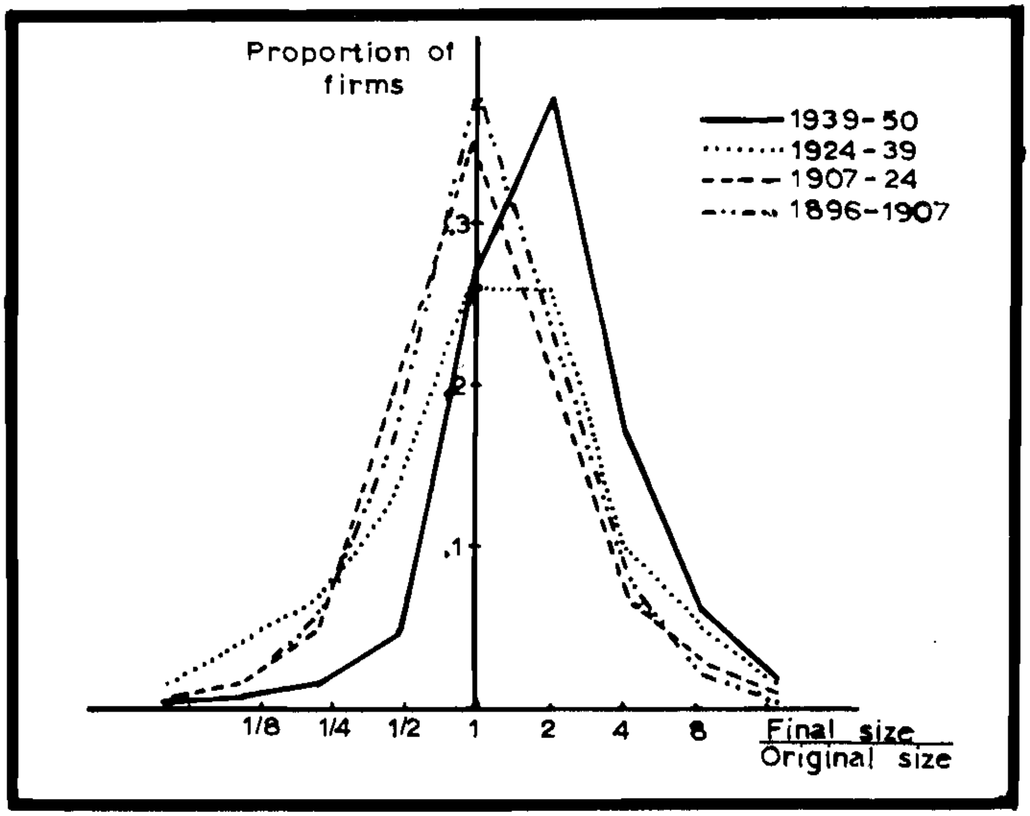
They grouped 1,939 British firms into 17 groups or categories according to the net assets possessed by each company and studied how the firms moved through the categories; in other words, the growth of the firms from periods of time between 1885 and 1950. Then they calculated from the data the probability of growth of each firm. (See Figures 1 and 2 for different periods of time.)

As it can be seen from the figures, the process of growth of the firms is described quite well by a log-normal distribution type function confirming the assumption of the proportionate growth.

From Figure 2 the average amount of growth as shown by the position of the maximum ordinate of each curve varies from period to period, but as the authors point out, this was due to the reflection of changes in prices. It is worthwhile to call to attention the similarity and near normality of dispersions of the curves over periods of time which had such different economic characteristics. This is the first model describing growth and size relationships of firms that appeared in the literature. Its influence over later works in this field has been remarkable, particularly in the use of the law of proportionate growth as an assumption in later models. Simon and Bonini (21) apply stochastic analysis to the construction of a model of growth and size distribution of firms in the United States. They sustain, as Bain (6), that the characteristic cost curve for a firm shows virtually constant return to scale for firm sizes above some critical minimum in a particular industry. Under the circumstances,



Growth Of British Firms During 1907 to 1939.
FIG 1



Probabilities Of The Growth Of British Firms 1896-1950
FIG 2

the static cost theory may predict the minimum size of firms in an industry, but it will not predict the size distribution of firms.

Simon and Bonini (21) proposed as theoretical approach for the explanation of growth and size distribution of firms a stochastic model based on the following assumptions:

(a) The size of the firms has no effect upon expected percentage growth. That is to say, we assume that a firm randomly selected from those of one million dollars in assets has the same probability of growing, say, 20 percent, as a firm randomly selected from those with one billion dollars in assets. The reasons for this assumption are that it agrees with the empirical findings of Hart and Prias (11) and also the assumption that there exists approximately constant return to scale above a minimum size of firm.

(b) There is a minimum size of firm--Sm--in an industry, and for firms above this size,unit costs are constant. Individual firms in the industry will grow or shrink at varying rates, depending on such factors as profit, dividends, policy, mergers, access to particular factors of production, market operations, etc. All these forces, Simon points out (21), will generate a probability distribution for the changes of firms of a given size.

(c) New firms are being "born" in the smallest size class at a relatively constant rate.

Under these hypotheses the authors prove that the steady-state distribution of this process is given by the Yule distribution (20).

Let f(s)ds represent the probability density of firms of size "s." Then the Yule distribution is given by

$$f(s) = KB(S, \rho + 1) \qquad (2\text{-}1)$$

where $B(S_1\rho + 1)$ is the Beta Function of s and $(\rho + 1)$, K is a normalizing constant, and ρ is a parameter.

One property of (2-1) is that when $S \to \infty$,

$$f(s) = MS^{-(\rho + 1)} \quad , \qquad (2\text{-}2)$$

which is the Pareto distribution.

The parameter ρ of the Yule distribution has the following interpretation: let G be the net growth of assets of all firms in consideration during some period, and let g be that part of the net growth attributable to new firms; i.e., firms that have reached the minimum size during the period. It can be shown that

$$\frac{1}{1-g/G} = \frac{1}{1-\alpha} \quad , \text{ where } \alpha = g/G \quad .$$

Thus, if $g/G = 0.1$, new firms account for 10 percent of the growth in assets in the industry. In the limit as the contribution of new firms to total growth approaches zero, ρ approaches 1.

Simon and Bonini (21) used the model to calculate the size distribution of the 500 largest firms in the United States in 1956, obtaining good results. A value of g/G was 0.23, or, 18.7 percent of the growth in assets of the American firms was accounted for by new firms.

Even more interesting, the model was used to determine the minimum efficient feasible plant size of some industries. The procedure was this: if there is a sharp increase in unit costs below some critical size, Pm, the number of plants in the industry below that size should be less than the number predicted from the Yule process. So they plotted cumulative numbers of plants against size on log paper and looked for sharp bends from a slope approximating -1 to a lower slope. (See Figure 3.) Census data for number of employees was used and converted to percent of total value added by the manufacturer.

In a later paper that could be considered as a continuation of the preceding, Simon and Ijiri (13) demonstrate the goodness of use of stochastic models to predict the size distribution of firms in a given industry. They observe the problems of testing statistically the goodness of fit of the stochastic models. In the case of the size distribution of firms, for instance, the observed distribution looks like log-normal Yule or Pareto distributions, but there is no known satisfactory way to objectify the degree of resemblence. Consequently, the confidence in the proposed stochastic explanation of size distribution may depend quite as much on how plausible we find the assumptions underlying the models as on the judgments of the goodness of fit. Even if the assumptions made are somewhat unrealistic, it is to be expected that the stochastic model will still give consistent results. To prove this hypothesis, Simon (13) assumes again the same stochastic model described in the preceding pages, but the first assumption is changed. The change in size of each firm is governed by a stochastic process, which depends not only on the size to which the firm has grown but also on the times at which its growth took place.

For simplicity they assume growth took place in increments of unit magnitude. The probability that a firm will experience an increment in

Industry	Bain Estimate as Percent of National Market	Estimate from Census Data by Yule Distribution as Percent of Total Value Added by Manufacture
Flour and milling	0.05 to 0.25	0.07 to 0.19
Footwear	no minimum	0.03 to 0.07
Canned fruits and vegetables	no minimum	0.06 to 0.11
Cement	0.4 to 0.7	0.14 to 0.54
Distilled liquors (except Brandy)	0.2 to 0.3	0.03 to 0.11
Petroleum refining	0.4 to 0.9	0.12 to 0.34
Meat packing	no minimum	0.3 to 0.7
Rubber tires and tubes	0.35 to 0.7	1.6 to 5.5
Rayon	1.0 to 3.0	0.14 to 0.37
Soap and glycerin	0.2 to 0.3	0.03 to 0.11
Cigarettes	1.0 or less	0.08 to 2.0
Fountain pens and mechanical pencils	1.3 to 2.5	0.06 to 0.16
Typewriters	5.0	5.7 to 14.1

Figure 3. Estimate of Minimum Feasible Plant Size (21)

size during the next time period is assumed proportional to a weighted sum of the increments it has experienced in the past, where the weight of an increment decreases geometrically at a rate β, with the lapse of time since its occurrence. In other words, let $X_j(\tau)$ be the change in size of the j^{th} firm during the t^{th} time interval, where $X_j(\tau)$ is either unity or zero. The firm either experiences a unit increment in size or remains the same size during any given time interval. Then the size of the j^{th} firm at the end of the t^{th} interval is simply

$$\sum_{\tau=1}^{t} X_j(\tau)$$

The expected increment in size of the j^{th} firm during $(t + 1)^{st}$ interval is

$$W_j = \text{Prob}\ [X_j(t + 1) = 1] = K(t) \sum_{\tau=1}^{t} X(\tau)\ \beta^{\tau}$$

where $K(t)$ is a function of time that is the same for all firms, and β is the fraction that determines how rapidly the influence of past growth on new growth drops out. It should be noted that this model does not admit mergers or decreases in size of individual firms. But as the authors sustain the probabilities of merger or declining size are roughly independent of the stratum, they will not change much the equilibrium distributions. Under these assumptions larger firms will grow proportionately more rapidly than small firms. On the other hand, firms that have experienced recent growth will grow more rapidly than firms of the same size whose growth took place earlier.

The model also assumes a constant probability α that new firms enter at an expected rate proportional to the rate of growth of the industry.

The running simulation of the model the authors found that the equilibrium distribution function the model generated closely resembled the Yule distribution, although the model incorporated a significant modification of the law of proportionate growth.

Simon and Ijiri (13) conclude that the behavior of the new model is consistent with empirical observations and exhibits a great serial correlation in the size changes of individual firms. Also, they point out "how a stochastic process that admits this very strong effect of recent growth in the determinants of future growth still produces the familiar kind of equilibrium distributions." (21). This is the reason for the plausibility of a stochastic explanation for observed distribution of firm sizes.

Hymer and Pashiagan (18), after an empirical study which covered 1,000 of the largest American firms in the period 1946 to 1955, used the size growth rate of the firms in an industry as variables to explain the firms' growth rate.

The size of firms was not significant as an explanatory variable, while the industry growth rate had a positive though weak effect; in other words, a confirmation of the law of proportionate growth.

Although no systematic relationship was found between the firm size and mean growth rate, an inverse relationship was observed between size and the standard deviation in growth rates. The smaller variability in the growth rates of large firms is due probably, they say, to their greater diversification. That is to say, they may be merely a collection of several smaller independent firms. If this happens the standard deviation of the growth rates of the larger firms would be $1/\sqrt{N}$ times the standard deviation of the growth rates of the smaller firms, where "N" is the number of firms required to make up a large firm. The predicted standard deviation on the assumptions of independence approximates the observed standard deviations for firms with less than 20 million dollars in assets. Beyond this size the actual standard deviations for firms exceeded the predicted standard deviations, indicating that the advantages of diversification diminish as size of firms increases.

T. Y. Shen (19) handles the problem of growth on size distribution of firms combining both stochastic process and economic theory. He states that firms have little tendency to become more concentrated in any particular size class. This can only mean that firms are not operating under conditions of long-run static equilibrium cost-curves and that the concept of optimal scale cannot be used to explain the actual size distribution and growth patterns of firms.

> If the stochastic models may give a satisfactory stationary distribution of the process describing growth of firms, this growth only can be explained by such factors as economies of scale, economies of growth (17), profit margin variations, and other dynamic factors. (19)

The appropriate frame of reference for an optimal distribution of size of firms, he says, is the expansion path or scale path. The path may be regarded as representing the basic manufacturing activity of an industry. Firms cluster around this path and their sizes are given by their positions on the path. The size distribution of firms is therefore defined with respect to this path. Similarly, from one period to the next the growth of the firms is given by their movement along the path.

Shen built a model to investigate the extent of economies of scale along the expansion path for some manufacturing industries and the relationships between economies of scale and the growth pattern of plants. The main hypothesis he made is that with a given expansion path--or given return to scale--all the plants tend to expand at the same rate. The strict form of the law of proportionate growth is assumed and also the log-normal distribution of firms is stated. He also assumes that when a shift in the expansion path takes place there is also a change in return to scale along the path. The change in return to scale results in a differential rate of growth for firms of different sizes, until a new equilibrium log-normal distribution is established. It should be noted that Shen's model produced good results when applied in the real world.

Adelman (1) faces the issue of the growth of firms in a given industry in a very original way. She says that the forces determining the growth and the distribution of size of firms within a particular industry are so varied and complex that any theoretical attempt to portray the effects of their interactions

> must be of necessity either drastically simplified or else hopelessly complicated. Therefore, it is better to assume that the growth pattern of firms is a size-dependent stochastic process with probabilities of transition constant in time. (1)

She assumes that the firms which comprise an industry are grouped according to some criterion of size into a number of classes, and also assumes that the growth or evolution of a firm through these classes is a stochastic process in which the probability per unit time of movement from one group to another is a function only of the two groups involved. That is to say, the probability that a firm will advance a given number of steps during a period depends only upon its size at the beginning of the period and the number of steps involved, and is independent of the previous history of the firm. Therefore, the growth of a firm is statistical in nature with absolute size as determinant of development. Under these circumstances, and using the theory of Markov chains (9) one arranges the transition probability into a square matrix upon a vector which represents the structure of the industry at the beginning of one period; it is possible to derive the structure of the next time interval. If the process is repeated indefinitely it leads to a vector which describes the equilibrium state.

In order to provide for entry into and departure from the industry, Adelman adds to the "n" size classes a large additional group which acts as a reservoir of potential entrants into the system, and assigns as the probability of moving from this zeroth group to, say, the jth group a value just significant enough to make the average number of firms entering the jth class per year correspond to the actual number of new firms started annually in the appropriate size range. Similarly, the failure of a firm will be represented as a movement into the zeroth class. Mathematically speaking, there is a transition probability matrix

$$[P] = \begin{vmatrix} P_{00}, P_{01}, \ldots, P_{0m} \\ P_{10}, P_{11}, \ldots, P_{1m} \\ \cdot \\ \cdot \\ \cdot \\ P_{m0}, P_{m1}, \ldots, P_{mm} \end{vmatrix}$$

in which all P_{ij} are non-negative and

$$\sum_{j=0}^{m} P_{ij} = 1 \qquad \text{for all } i \ (9)$$

where P_{ij} denotes the probability that a firm in size class "i" will enter the group "j" during the next period. For instance, P_{20} will stand for the probability that during a unit of time a firm in size range "2" will go out of business for any reason whatsoever. [P] is a stochastic matrix so it may be assumed that all the states are accessible and that an equilibrium solution to the process exists (9).

Symbolically, if the structure of an industry is described by a row vector

$$(S_j^n) = (S_0^n, S_1^n, S_2^n, \ldots, S_m^n)$$

where the components of which represent the proportion of firms in each class at time "n", the configuration after the next step may be found from

$$S_j^{n+1} = (S_j^n)\ [P] \quad , (9)$$

and by successive substitutions one may write

$$(S_j^o)\ [P]^{n+1} = S_j^{n+1} \quad ,$$

and the equilibrium vector could be found by multiplying [P] times itself a large number of times. However, a simpler approach is to make use of the fact that in equilibrium the distributions of firms among the statum must be invariant. For the equilibrium vector, t_j, the following condition is observed:

$$(t_j) = (t_j)\,[P] \quad , \quad (9)$$

where t_j represents a relative distribution of the size of the firms in a given industry. So:

$$\sum_{j=0}^{m} t_j = 1 \quad .$$

The mean life time L_j of a corporation in the j^{th} stratum may be calculated by noting that the total time spent in an interval by all the S_j firms originally included therein is given by

$$T_j = S_j^o + S_j^o\,P_{jj} + S_j^o\,P_{jj}^2 + \ldots$$

Therefore, the original firm will remain in the j^{th} level for a period.

$$L_j = \frac{T_j}{S_j^o} = 1 + P_{jj} + P_{jj}^2 + \ldots = \frac{1}{1-P_{jj}} \quad .$$

The transition probabilities P_{ij} can be calculated as the author suggests. If A_{ij} represents the number of movements of firms from class "i" to class "j" through the period under consideration, our transition probabilities P_{ij} become

$$P_{ij} = \frac{A_{ij}}{\sum_{j=0}^{m} A_{ij}}$$

for all i. The author uses the model to predict the structure and growth of the steel industry with satisfactory results.

Using a similar approach, Collins and Preston (7) studied the structure of the 100 biggest companies in the United States and the growth of the food processing industry in the period 1935 to 1955 (8). Finally, Mansfield (15) used stochastic processes in the petroleum and rubber industry.

Biological Models for Social Organizations

The search for an understanding of the growth of organizations not only has taken its bases on economic and social principles, but on the similarities with living organisms. Particularly systems theory approaches have given place to this way of studying the growth of the firms. As we know, systems theory tries to look for general phenomena which can be found in many fields and seeks to build up a general theoretical model relevant to these phenomena (3).

Using this approach, many investigations have tried to take the well-known and more or less studied problem of growth of living organisms and see if we can apply them in similar fashion to social organisms (4). There is an extensive literature in this field, most of it trying to explain some analogies between the firm and the living organisms.

Marshall (22), probably the best known in this matter, assimilates the life cycle in which appearance, growth, and disappearance of firms are likened to the processes of birth, growth, and death of biological organisms.

More recently the viability analogy (2) and the homeostasis analogy (3) have been presented and designed to explain the behavior of the firm. Both are supposed to represent improvements on the existing theory of the firm at the core of which lies the chief target of attack: the assumption that the firm attempts to maximize profits.

With the increasing concern for problems of change, growth and development, investigators have turned to biology, the science most concerned with development, as a source of appropriate models to apply to social organizations. The growth model for social organizations can be formulated by looking for lawful process in organizational growth grounded in factors inside the forms and forces shaping it as it grows. They have also restated in specific terms the interdependence of size, shape, and function of an organization and found through empirical data how the organization has grown, thus determining the balance between the firm and its environment (12). Based on this statement, Mason Haire (12) proposed a biological model as means of understanding the growth and form of the firm.

The System Approach

In dealing with growing organizations we can see that limitations come from the size and shape function. As the organization grows its internal shape must change. Additional functions of coordination, control, and communication must be provided. Using the similarity with a living organism we can see that as the size of the firm increases the skeletal structure needed to support it against the forces tending to destroy it grows faster than the size of the firm itself. As this situation will consume a disproportionate amount of the productive capacity of the organization, it is important to identify the skeletal support of the firm, the forces it resists, and the rates at which the support must grow (12).

The way Haire faced this problem was by studying the history of growth of several firms and inferring the operating forces from the direction of changes in shape and function of size change.

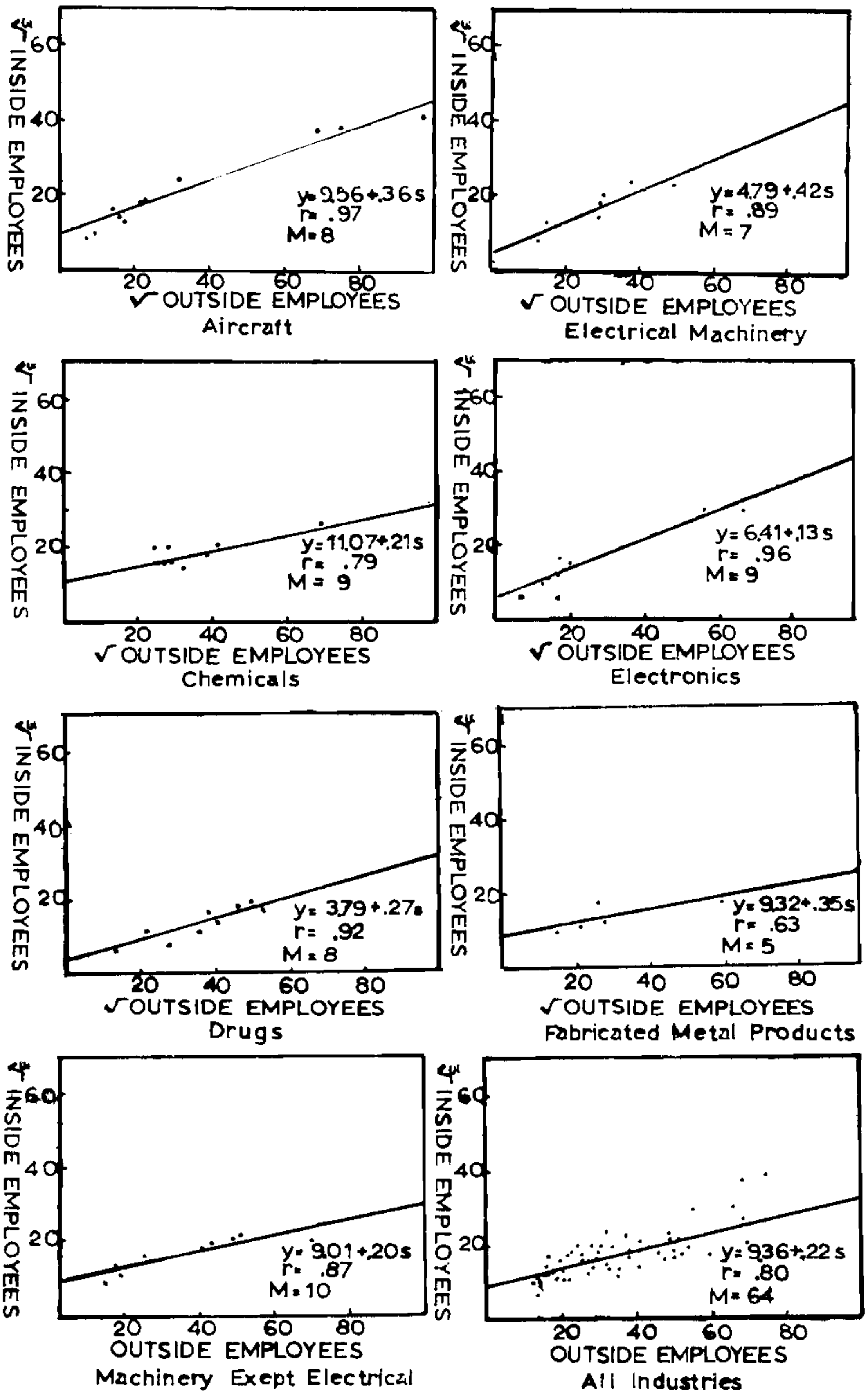
Haire proposed the "square-cube" law as a description of the growth of the firms. Such a law has been used as an application of the environmental forces to organisms. The "square-cube" law states that the mass grows by a cube function while surface grows by a square function.

If one were to take the cube root of the volume and the square root of the area and plot them for different stages of growth, the result would be a straight line from the origin with a slope of 1. As the cube root of the mass doubles, the square root of the surface doubles. Haire measured surface and volume of the industrial firm by the number of people who inhabit these two areas. Volume in this sense refers to employees dealing with the firm's internal functions, e.g., accountants. Surface refers to employees dealing with the firm's external environment, e.g., sales, purchasing. Haire studied longitudinal data on four firms and discovered that the plots of the cube root of the volume and the square root of the surface indicated remarkably high correlation coefficients: 0.99, 0.95, 0.96, and 0.97. Two deviations of the theoretical cube-square law were noticed. First, that the interception of the ordinate allows for zero surface and some mass, and impossibility in physical geometry. Second, that the slopes for this case were all positive, ranging from 0.50 to 0.97, but fell short of the unity predicted by the cube-square law. Haire concludes that these deviations are perhaps due to the arbitrary definitions of mass and surface, or because a special characteristic applying to social organisms exists and the principles do not apply rigorously. However, the regularity with which the data fit a simple expression derived from a geometrical model strengthened the possibility of an eventual representation of social groups in a useful geometry allowing changes in shape and growth to be seen, and from this, making it possible to infer the size and direction of the forces associated with organization expansion.

Continuing the work of Haire, Levy (14) applied the cube-square law to analysis of the aircraft, chemical, drug, food, electric, and electronics industries. He states that the linearity of the cube-square law found by Haire suggests that the environment for any firm remains relatively constant in terms of the organizational pattern that must be maintained or extended in order to be successful in that environment. If this is the case, then a cross-section analysis of the firms belonging to reasonably homogeneous industry groupings which share a common environment should similarly display a linear relationship between the cube root of the volume (inside employees) and the square root of the surface (outside employees).

Levy (14) analyzed data obtained on sixty-two firms in nine industry groups. Each occupational category was assigned either inside-volume or outside-surface of the firm. For example, marketing, procurement, and receptionist employees deal with the external environment. Conversely, research, personnel, and production functions are inside. The results obtained by Levy are shown in Figure 4.

Levy concludes that the cube-square law is a reasonable and consistent description of the industrial organizational growth among firms of varying size in different industries. It is clear that the relationship of inside to outside personnel varies significantly among industries,



Pattern Of Growth Of Some American Industries

FIG 4

and this variance in turn is a reflection of environmental differences and the effects of such environmental differences on organizational composition.

Finally, Levy was able to isolate one characteristic of industries in relation to the cube-square law regression lines slope, namely, a positive correlation with the proportion of total staffing engaged in research.

CHAPTER III

THEORETICAL MODELS OF SIZE DISTRIBUTION OF FIRMS REGARDED AS A STOCHASTIC PROCESS

The forces determining the distribution of firm-sizes in any industry are so varied and complex, and interact and fluctuate so continuously that any theoretical approach must either be unrealistically simplified or hopelessly complicated. We shall choose in our work the former alternative, but then give indications that the introductions of some of the more obvious complications of the real world do not seem to disturb the general trend of our conclusions. We shall suppose that the firms which comprise an industry can be grouped according to some criterion of size into an enumerable infinity of size classes or ranges. For example, we might consider as criterion for measuring the size of a firm the dollar value of the total assets of the given firm and choose as size-range one million-two million, two million-four million, four million-eight million, etc., although a finer gradation would be more interesting.

We shall regard the development through time of the distribution of the firm's size as being a stochastic process so that the size of any firm in one year may depend on what it was in the previous or base year and in a chance process.

Under such assumptions any development of the distribution of firm sizes could be described in terms of the following vectors and matrices: $X_r(0)$, telling us the number $X_r(0)$ of firms in each range R_r, $r = 0, 1, 2, \ldots$, in the initial year Y_o, and a series of matrices $P_{rs}(t)$ telling us in each year Y_t the proportion of occupants of R_r who are shifted to range R_s in the following year $Y_t + 1$. With these definitions the size distribution $X_r(t)$ in the successive years will be generated according to

$$X_s(t + 1) = \sum_{r=0}^{\infty} X_r(t)\ P'_{rs}(t) \quad , \tag{3-1}$$

or in more general form

$$X_s(t + k) = \sum_{r=0}^{\infty} X_r(t)\ P'_{rs}(t) \quad , \tag{3-2}$$

where $P'_{rs}(t)$ in this case indicates the proportion of the occupants in R_r who are shifted to range R_s in the next k years.

If we suppose that the firm's size ranges are paired in order of size--there being a lowest firm-size range R_o--then there will be some advantage in defining a new set of matrices.

$$P_{r\mu}(t) = P'_{r,r+\mu}(t) \quad , \tag{3-3}$$

and rewriting (3-1) in the form

$$X_s(t + 1) = \sum_{\mu=-\infty}^{s} X_{s-\mu}(t)\ P_{s-\mu,\mu}(t) \quad , \tag{3-4}$$

where $P_{r\mu}(t)$ is the proportion in Y_t of firms in R_r that shift up by various numbers μ of ranges.

The advantage arises from the fact that in the real world the magnitude of such shifts from year to year is, mostly, fairly limited so that each $P_{r\mu}(t)$, regarded as a frequency distribution in μ, is likely to be centered on $\mu = 0$. In order to simplify this model we should like to be able to assume that the $P_{r\mu}(t)$, regarded as a frequency distribution in μ, differed very little in form for variations over a wide range of values of r and t. In other words, that the prospects of shifts upwards and downwards along the ladder of size ranges differs little between the occupants of different size ranges and differs little from year to year.

This obviously cannot apply to all size ranges. For example, it is doubtful that a big firm, unlike a small firm, will be degraded to a lower size range in the following year.

Again the absolute changes in size are likely to be much higher for firms whose total assets are worth one billion dollars than for firms whose total assets are worth ten million dollars, so that the ranges of firm size must have a greater absolute magnitude for big than for small size firms, if our simplification is to have any plausibility.

The choice of size-ranges in which we are going to group the different firms will be made in such a way that each range has equally proportionate ranges; for then any effects, such as price or market movements, which we suppose are likely to alter size prospects for widely different ranges R_r and R_g in approximately the same manner, proportionately, will affect the various functions $P_{r\mu}(t)$ and $P_{g\mu}(t)$ in roughly the same fashion.

This assumption of size-firm ranges follows the spirit of Adelman (1) and Hart (11). We will also assume that the functions $P'_{r,r+\mu}(t) = P_{r,\mu}(t)$ remains constant as t changes through time. Of course such assumption takes us far from reality; but it is analogous to that used in long-run comparative statics; that the forces which operate during a period will continue unchanged until equilibrium is reached. It is assumed that if the time period over which the transition probabilities are evaluated is sufficiently long, the use of the approximation may be expected to give satisfactory results.

Under these conditions the development of the distribution of firm-sizes in a given industry can be described by a stochastic process involving infinite matrices, $P'_{rs}(t)$. Considerable interest may therefore be found in the question of the type of firm-size distribution which will correspond to the repeated operation of the changes represented by any realistic form of the matrix $P_{rs}(t)$.

It would be a great advantage in constructing models of firm-size distribution in an industry if we had empirical evidence about matrices $P'_{rs}(t)$ describing actual movements of the firms in modern industries.

A Model Generating a Pareto Distribution

If some considerations are made with respect to the matrix $P'_{rs}(t)$ we could obtain as the steady-state of the distribution of sizes of the firms in a given industry the Pareto Distribution. Although the assumptions of this section do not completely approach reality, the results they lead to resemble reality in one respect, and this will help in understanding this aspect of actual distributions of firm sizes.

Let us assume that for every value of t and r and for some integer n

$$P'_{rs}(t) = P'_{r,r+\mu}(t) = P_{r,\mu}(t) = 0 \quad , \qquad (3\text{-}5)$$

if $\mu > 1$ or $\mu < -n$. This means that no firm moves up by more than one size range in a year, or down by more than n firm-size ranges in a year. So

$$P'_{r,s}(t) = P'_{r,r+\mu}(t) = P_{r,\mu}(t) = P_{\mu} > 0 \quad , \tag{3-6}$$

if $-n \leq \mu \leq 1$ and $\mu > -r$. Equation (3-6) means that the prospects of shifts upwards and downwards on the ladder of firm-size ranges are distributed in a manner independent of the present firm's size, apart from the limitations imposed by the impossibility of falling below the bottom rung of the ladder.

To accomplish mathematical requirements we assume that for each value of r and t

$$\sum_{s=0}^{\infty} P'_{r,s}(t) = \sum_{\mu=-r}^{\infty} P'_{r,r+\mu}(t) = 1 \quad . \tag{3-7}$$

Equation (3-7) implies that

$$\sum_{\mu=-n}^{1} P_{\mu} = 1 \tag{3-8}$$

for all r. Or, in other words, we assume a stationary process or that the transition probabilities do not change through time.

Another assumption should be made in order to ensure that the process is not dissipative (16): that the transition matrix has a limit, or in other words, that the size distribution of the different firms settle down to an equilibrium distribution. Mathematically, this condition is ensured by

$$\sum_{\mu=-n}^{1} P_{\mu} Z^{1-\mu} = Z \quad . \tag{3-9}$$

This completes the list of assumptions for our model. As an example, when n = 3, they give rise to a matrix of Figure 5.

Now we should study the equilibrium distribution corresponding to any matrix $P'_{r,r+\mu}(t) = P_{r,\mu}(t)$ conforming to our assumptions.

According to the properties of a stochastic process, it will be sufficient to find any distribution which remains exactly the same under repeated action of the matrix $P'_{r,s}(t)$.

The assumptions (3-5) to (3-9) are only for the purpose of making the solution feasible and obvious. Indeed, if X_s is the desired equilibrium distribution, we need by (3-4), (3-5), and (3-7)

$$X_s = \sum_{\mu=-n}^{1} P_\mu X_{s-\mu} \tag{3-10}$$

for all s > 0, and

$$X_o = \sum_{-n}^{0} g_\mu X_{-\mu} \text{ where } g_\mu = \sum_{\nu=-n}^{\mu} P_\nu \quad . \tag{3-11}$$

Equation (3-10) is an equation in differences (10). The only solution of (3-10) for X_s--because by satisfying (3-5), (3-6), (3-7), and (3-10) we ensure the satisfaction of (3-11)--is a solution of the form

$$X_s = b^s \quad , \tag{3-12}$$

where b is a positive real number other than unity obtained equating to zero equation (3-9), or

s =	0	1	2	3	4	5	6	7	8	.	.
r =	$P'_{rs}(t)$										
0	$1 - P_1$	P_1	0	0	0	0	0	0	0	.	.
1	$1 - P_0 - P_1$	P_0	P_1	0	0	0	0	0	0	.	.
2	$P_{-3} + P_{-2}$	P_{-1}	P_0	P_1	0	0	0	0	0	.	.
3	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	0	0	0	0	.	.
4	0	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	0	0	0	.	.
5	0	0	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	0	0	.	.
6	0	0	0	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	0	.	.
7	0	0	0	0	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.

Figure 5. Stochastic Matrix for n = 3

$$F(Z) = \sum_{\mu=-n}^{1} P_{\mu} Z^{1-\mu} - Z = 0 \quad . \tag{3-13}$$

Analyzing (3-13) one root is obvious, $Z = 1$, and using Descartes' Rule of Signs, since when $F(0) = P_o > 0$, the other real positive root must satisfy

$$0 < b < 1 \quad . \tag{3-14}$$

Hence, the solution (3-12) implies that the total number of firms is given by

$$N' = \frac{1}{1 - b} \quad . \tag{3 15}$$

So if we want to arrange for any given number N of firms in an industry we need merely modify (3-12), hence

$$X_s = N(1-b)\ b^s \quad . \tag{3-16}$$

In other words, if we multiply (3-12) by a constant, the solution is still valid. Now suppose that the proportionate extent of each firm-size range is A^h, and that the lowest firm size is Y_{min}: then X_s is the number of firms in range R_s whose lower bound is given by

$$Y_s = A^{sh} Y_{min} \quad , \tag{3-17}$$

or taking logarithms in base A

$$\log_A y_s = sh + \log_A y_{min} \quad . \tag{3-18}$$

By adding a geometrical progression using (3-16) we will find that in the long run, or mathematically speaking, in the steady-state the distribution of the numbers of firms exceeding Y_s is given by

$$F(Y_s) = Nb^S \quad , \tag{3-19}$$

whence

$$\log_A F(Y_s) = \log_A N + S \log_A Y_{min} \quad . \tag{3-20}$$

Let $\alpha = \log_A b^{-1/h}$ and $\gamma = \log_A N + \alpha \log_A Y_{min}$. Then it follows from (3-17), (3-18), and (3-19)

$$\log_A F(Y_s) = \gamma - \alpha \log_A Y_s \quad . \tag{3-21}$$

This means that for $Y = Y_0, Y_1, Y_2, \ldots$, the logarithm of the number of firms exceeding Y is a linear function of Y. This states the Pareto's Law in its exact form (20)(24).

Thus, if all ranges are of equally proportionate extent, our simplifying assumptions ensure that any initial distribution of firm's size will in the long run approach the exact Pareto distribution given by (3-20) and (3-21). Bonini (21) fitted the Yule distribution to the five hundred largest American firms. Hart and Prias (11) applied a log-normal distribution as the distribution function of the sizes of firms in British industry. It can be proved that when the parameters of these distributions go to infinity (i.e., when they reach steady-state) both distributions have as a limit the Pareto distribution (20).

The model presented here brings out clearly the tendency for

Pareto's law to apply in an industry where, above a certain minimum size, the prospects of various amounts of percentage change in the size of the firms are independent of their initial size.

Although the assumptions upon which the model is based are not very realistic, the results obtained do not contradict the empirical findings on this subject, as we shall see later on. For this reason we think the model can be applied in real world situations.

Some modifications of the model should be mentioned:

1. By removing the assumption that there is a lowest size range R_o, and setting up adequate conditions, it could be possible to obtain a two-tailed distribution: one for the biggest companies and the other for the little and medium companies in the industry.

2. More generalization is obtained if we allow the different firms to shift upwards by more than one range during the period of time in consideration.

3. By limiting our basic assumption that the prospects of various amounts of percentage change of size are independent of the initial size to apply especially to the big companies as it happens in the real world.

From these three modifications to our basic model the most straightforward is the second one.

If we eliminate the restrictive assumption that $P_{r\mu}(t) = 0$ when $\mu > 1$ and $r \geq 0$, let us replace the assumption (3-5) by

$$P'_{rs}(t) = P'_{r,r+\mu}(t) = P_{\mu} = 0 \quad , \qquad (3\text{-}22)$$

if $\mu > m$ (m a given positive number) or $\mu < -n$, and (3-6) by

$$P'_{rs}(t) = P'_{r,r+\mu}(t) = P_\mu \tag{3-23}$$

(defined for $\mu = -n, 1-n, \ldots, m$ and all r).

We may retain assumptions (3-7) and (3-8) which in this case turn out to be

$$\sum_{\mu=-n}^{m} P_\mu = 1 \qquad \text{with all } P_\mu \geq 0 \tag{3-24}$$

$$\sum_{\mu=-n}^{m} P_\mu Z^{m-\mu} = Z^m \quad . \tag{3-25}$$

In the case that $n = 2$ and $m = 3$, the matrix $P'_{rs}(t)$ will have the form shown in Figure 6. The required solution will be

$$\begin{aligned} X_s &= \sum_{r=0}^{\infty} P'_{rs}(t)\, X_r \qquad s = 0, 1, 2, \ldots \\ N &= \sum_{s=0}^{\infty} X_s \quad . \end{aligned} \tag{3-26}$$

Equation (3-26) could be replaced by

$$X_s = \sum_{\mu=-m}^{m} P_\mu X_{s-\mu} \qquad s = 0, 1, 2, \ldots \quad . \tag{3-27}$$

Again, (3-27) is a difference equation similar to (3-10) with the difference being that (3-27) is of higher order $(m + n)$.

A solution of (3-27) is of the form

r =	0	1	2	3	4	5	6	7	.	.
s = 0	$1 - P_{-2} - P_{-1} - P_0$	P_1	P_2	P_3	0	0	0	0	.	.
1	$P_{-2} + P_{-1}$	P_0	P_1	P_2	P_3	0	0	0	.	.
2	P_{-2}	P_{-1}	P_0	P_1	P_2	P_3	0	0	.	.
3	0	P_{-2}	P_{-1}	P_0	P_1	P_2	P_3	0	.	.
4	0	0	P_{-2}	P_{-1}	P_0	P_1	P_2	P_3	.	.
5	0	0	0	P_{-2}	P_{-1}	P_0	P_1	P_2	.	.
6	0	0	0	0	P_{-2}	P_{-1}	P_0	P_1	.	.
7	0	0	0	0	0	P_{-2}	P_{-1}	P_0	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

Figure 6. Stochastic Matrix for $m = 3$, $n = 2$

$$X_s = \sum_{k=0}^{m+n} B_k b_k^{\,s} \qquad s = 0, 1, 2, \ldots \quad . \tag{3-28}$$

Since no X_s can be negative, the b_k's must be roots of (3-25), and since all the coefficients of non-zero powers of Z in (3-25) are positive, (3-28) may be reduced to $X_s = Bb^s$ where b is the real root lying between zero and one, and B a positive constant. Following a reasoning similar to the previous model we obtain again the Pareto distribution as steady-state distribution of the process.

<u>A Model Making Allowances for Some Effects of Technology Improvements</u>

An obvious objection to the preceding models is the assumed constancy over time of the movements matrices $P_{rs}(t)$ over time. It is a fact that the technological advances in all the fields of science play an important role in determining, in part, the size of firms in the different industries. In this section we will try to modify our previous models in order to consider in some way the degree of technology the different firms have reached as a factor influencing their growth and consequently their sizes in a given industry.

Our methods will be to suppose that our industry in consideration--or population, statistically speaking--is divided into C "degrees of technology"; firms with high technological advances, firms with medium degree of technology, and firms with low technological achievements--and that the different firms can move from one degree of technology to another; the prospects of change of size varying from one degree of technology to another.

Let us now set down formally the notation for our original model, modified to include the technological aspect. Let $P'^{cd}_{rs}(t) = P^{cd}_{r,s-r}(t)$ denote the proportion of the firms in a given industry in range-size R_r in degree of technology C in year t, which move into range R_s in degree of technology d in year t + 1.

If $X^c_r(t)$ denotes the number of firms in size-range R_r and degree of technology c in year t, then by definition

$$X^d_s(t) = \sum_{c=1}^{C} \sum_{r=0}^{\infty} P'^{cd}_{rs}(t)\ X^c_s(t) \quad , \tag{3-29}$$

and if an equilibrium distribution X^d_s exists it must satisfy the condition

$$X^d_s = \sum_{c=1}^{C} \sum_{r=0}^{\infty} P'^{cd}_{rs}(t)\ X^c_r \quad , \tag{3-30}$$

for every d = 1, 2, . . . C, and s = 0, 1, 2, ... The assumptions for our model can now be set down in a form closely analogous to those of our previous model. We assume

$$P^{cd}_{rs}(t) = P^{cd}_{r,r+s}(t) = 0 \quad , \tag{3-31}$$

when $s > m_d$ or $s < -n_d$.

Where m_d and n_d are positive integers for d = 1, 2, 3, . . . c

$$P^{cd}_{r\mu}(t) = P'^{cd}_{r,r+\mu}(t) = P^{cd}_{\mu} \geq 0 \tag{3-32}$$

(defined for values of $\mu = -n_d, 1 - n_d, \ldots, m_d$) and P^{cd}_{μ} are constants satisfying for each C = 1, 2, . . ., C the condition

$$\sum_{d=1}^{c} \sum_{\mu=-nd}^{m_d} P_{\mu}^{cd} = 1 \quad . \tag{3-33}$$

It is also convenient at this point to make some remarks:

1. It is necessary to clarify the accessibility of one size-range R_s in one degree of technology C_d from the size range R_r in a degree of technology C_c. We will assume a perfect mobility through all the size-ranges in the different degrees of technology (9). For instance: range R_s in C_d will be called accessible in one step from range R_r in C_c, if $P'^{cd}_{rs}(t)$ is positive. In general, we will say that R_s in C_d is accessible in n steps from R_r in C_c if it is accessible in one step from any range size in any degree of technology which is itself accessible from R_r in C_c in n-1 steps.

Finally, R_s in C_d will be termed accessible from R_r in C_c if for any n it is thus accessible in n steps.

2. In order to ensure that the size distributions of firms is unique (9)(16), each size-range in any degree of technology is accessible from each size-range in every degree of technology. Mathematically, this is what we have stated in equation (3-33), or more generally,

$$\sum_{d=1}^{c} \sum_{\mu=-n_d}^{m_d} P_{r\mu}^{cd}(t) = 1 \quad . \tag{3-34}$$

In this condition a generalized model of this type shall have C sets of equilibrium equations to be satisfied by the equilibrium distribution X_s^d, namely

$$X_s^d = \sum_{c=1}^{C} \sum_{r=0}^{\infty} P'^{cd}_{rs}(t)\, X_r^c \tag{3-35}$$

for d = 1, 2, . . ., C, or taking into account (3-31) and (3-32)

$$X_s^d = \sum_{c=1}^{C} \sum_{\mu=-n_d}^{m_d} P_\mu^{cd}\, X_{s-\mu}^d \quad \text{for } d = 1, 2, \ldots, C. \tag{3-36}$$

We are led to investigate C simultaneous equations:

$$A_d = \sum_{c=1}^{C} \sum_{\mu=-n_d}^{m_d} A_c P_\mu^{cd} Z^{-\mu} \quad d = 1, 2, \ldots, C \tag{3-37}$$

which may be written again as

$$\sum_{c=1}^{c} A_c P^{cd}(Z) = 0 \qquad d = 1, 2, 3, \ldots, C \tag{3-38}$$

where

$$P^{cd}(Z) = \sum_{\mu=-n_d}^{m_d} P_\mu^{cd} Z^{-\mu} - 1 \quad \text{if } c = d \ , \tag{3-39}$$

or

$$P^{cd}(Z) = \sum_{\mu=-n_d}^{m_d} P_\mu^{cd} Z^{-\mu} \quad \text{if } c \neq d \ . \tag{3-40}$$

If (3-38) has a solution eliminating the coefficients, A_c implies

$$\text{Det} \left| P^{cd}(Z) \right| = G(Z) = \text{say} = 0 \quad . \tag{3-41}$$

The function G(Z) can be expressed in the form

$$G(Z) = \sum_{\mu=-n}^{m} P_\mu Z^{-\mu} \; , \; m = \sum_{c=1}^{C} m_c \; ; \quad n = \sum_{c=1}^{C} n_c \quad , \tag{3-42}$$

and G(Z) plays a role similar to equation (3-25). Then it can be proved that for large s, where X_s is the equilibrium distribution

$$X_s = Kb^s \quad , \tag{3-43}$$

where b is a real positive root of (3-42).

Then, again, the Pareto distribution is the steady-state distribution of the model.

CHAPTER IV

EVALUATION OF THE MODELS

In this chapter we shall illustrate the procedure of the preceding sections by applying the assumptions and tools developed previously to the derivation of the Pareto size distribution function of the firms in the computing accounting industry.

Generalities

For the purpose of this study, since there are no criteria for defining an industry which is free from abritrariness, it seemed sensible to use definitions usually accepted. The sample chosen represents an industry according to the 4-digit S.I.C. classes, specifically, the computing accounting industry which is labeled according to the S.I.C. classification with the number 3571.

The size of the firms was measured in terms of assets. The Dunn and Bradstreet Million Dollar Directory was the source of information for firms in the computing accounting industry. The actual assets figures were procured from the various relevant Moody's Manuals (23). If a given firm listed in Dunn and Bradstreet was not listed in Moody's it was eliminated from the sample. Similarly, we eliminated firms for which assets were not listed in Moody's or which assets were listed as subsidiaries of some parent company and for which no separate balance sheet was provided. These restrictions tend to eliminate, in our opinion, the small firms. However, although we are basically interested in the size distribution of the biggest companies of a given industry, the selection procedure is thus not free of arbitrariness. The particular industry chosen was selected mainly for two reasons: first of all, the computing and accounting industry is actually the most progressive and one of the most important industries in the nation's economy; secondly, the present market structure within this field of activity is fairly typical of that of industry in general. For as is well known, the current form of organization of the computing accounting industry is oligopolistic in character, consisting of a few dominant firms together with a large number of small enterprises.

Data and Distributions

In order to derive the shape predicted by our model for the equilibrium size distribution of firms of the computing accounting industry, a first step in this direction is the quantification of a matrix similar to the one in Figure 7.

Before we could accomplish this task we had to select the different size-ranges R_r through which firms are distributed.

To avoid the problem of statistical deflation--for it appears unreasonable to assume that a firm with ten million dollars worth of assets in 1963 is necessarily equal in magnitude to a ten-million dollar firm in 1965--we required that the class limits of a given size range represent the same share of the total assets of the firms in consideration in any specified year. In other words, we used relative sizes.

The years selected for our study were 1963-1965. Since the ratio of the value of the total assets of the industry in 1965 to their value in 1963 was approximately 1.5, arbitrarily we establish the upper limit of the

Size of Firm in 1963*	s = r =	Size of Firm in 1965: 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
.045 - .066	0	1	2																	
.067 - .100	1	1	1																	
.101 - .151	2		2	2		2		1												
.152 - .227	3			3	1															
.228 - .340	4																			
.341 - .511	5					1	1													
.512 - .768	6							2												
.769 - 1.151	7								1											
1.152 - 1.729	8								1			1								
1.730 - 2.58	9									1										
2.59 - 3.88	10										1									
3.89 - 5.83	11											1								
5.84 - 8.75	12													1						
8.76 - 13.11	13													2	1					
13.12 - 19.74	14																			
19.75 - 29.59	15																			
29.60 - 44.39	16																		1	
44.40 - 66.59	17																			
66.60 - 100	18																			

Figure 7. Cross-Classification of the Computing Accounting Industry by Their Relative Sizes in 1963 and 1965.

*A firm's size is here defined as its share of the total assets of all the firms in the specified year. Thus, the size classes here shown refer to the percentages of total assets.

largest size class as being 100 percent, and determine successive classes dividing each upper class by 1.5. Hence, the structure of possible size classes is determined as 100 - 66.6 percent, 66.59-44.5 percent, 44.49-29.7 percent, etc. Taking R_0 the range 0.067-0.101 percent, we develope a transitional matrix showing the number of transitional movements of firms from one size class to another over the period 1963-1965.

One problem in the construction and interpretation of this matrix should be noted. The establishment of size classes is necessarily arbitrary, and is significant because the size classes used determined the amount of movement recorded in the matrix. The procedure adopted here has been to construct size classes in terms of the share of each firm in the assets of the group at each point in time, and to determine arbitrarily that a firm must move from one size class to another when its relative share of the total assets either increases by 50 percent or falls by 33 percent.

An examination of the matrix reveals several interesting facts. First of all, the most probable outcome is that a firm will move one interval down or remain in the same class interval. Second, the firms move up and down by no more than one asset range. Of course this fact is weakened by the way we select our size intervals. It would require an extraordinarily high rate of growth for a firm to move from class j to class j + 2 in two years. Nevertheless, three companies did it.

Approximately 42 percent of the firms in consideration moved one class interval down; 35.5 percent stayed in the same class interval; 9.6 percent moved one size class up; and 12.9 percent moved by more than one size interval up.

Taking these percentages to represent an estimate of the probabilities that such shifts will occur in each of an indefinite number of subsequent periods, we proceed to evaluate the equilibrium size distribution predicted by our model for the computing accounting industry. Since all the firms except one moved by one size range down or two size ranges up, making in equations (3-22), (3-23), $n = 1$; $m = 2$, $P_{-1} = 0.42$; $P_0 = 0.36$; $P_1 = 0.096$; $P_2 = 0.124$, equation (3-25) becomes

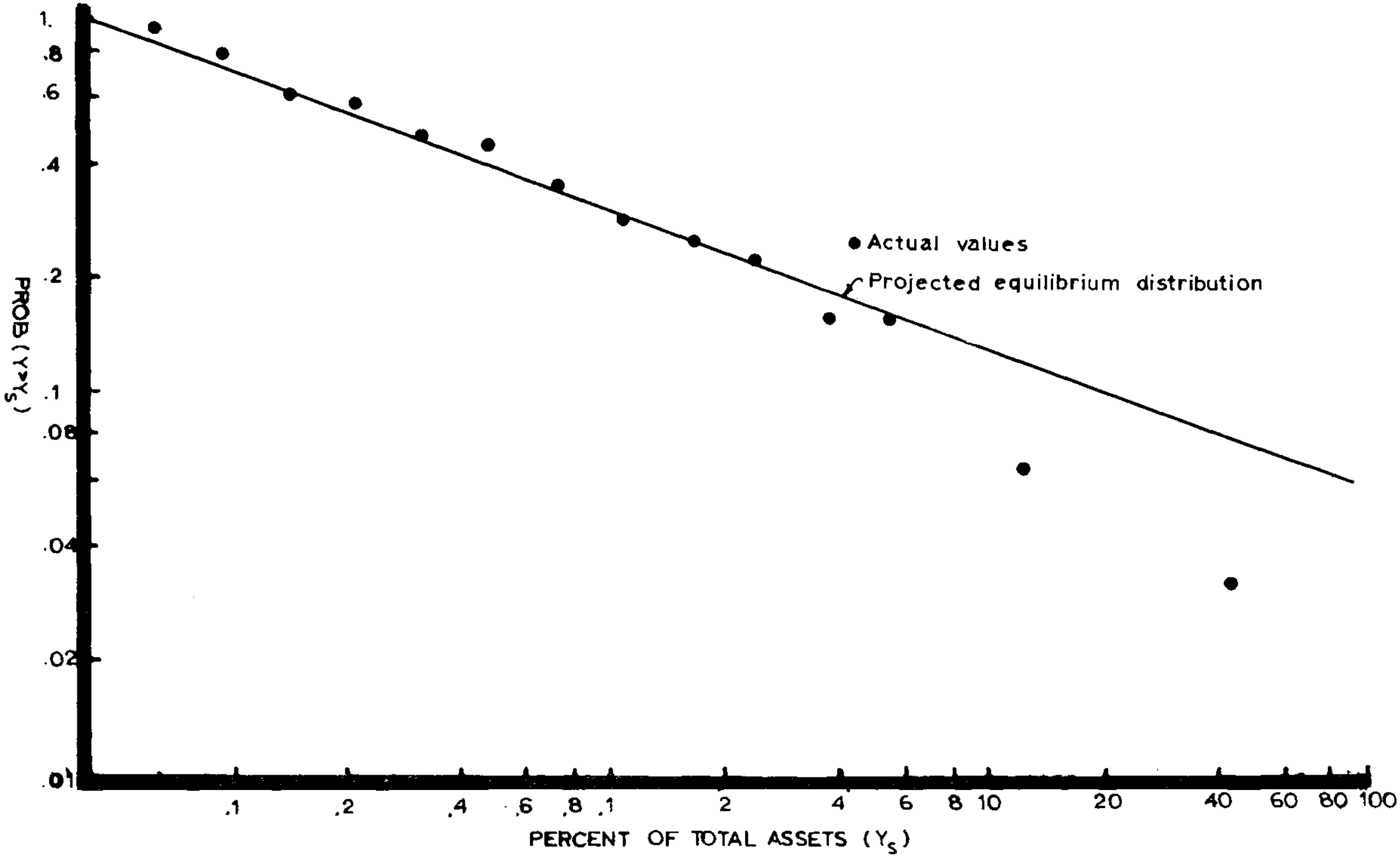
$$.42\, Z^3 + .36\, Z^2 + .096\, Z + .124 - Z^2 = 0 \quad , \qquad (4\text{-}1)$$

which gives a value of $b = 0.865$. Using this value in equations (3-20) and (3-21), the Pareto distribution of firm sizes in the computing accounting industry is given by

$$P(Y > y_s) = \left(\frac{.045}{y_s}\right)^{0.36} \quad , \qquad (4\text{-}2)$$

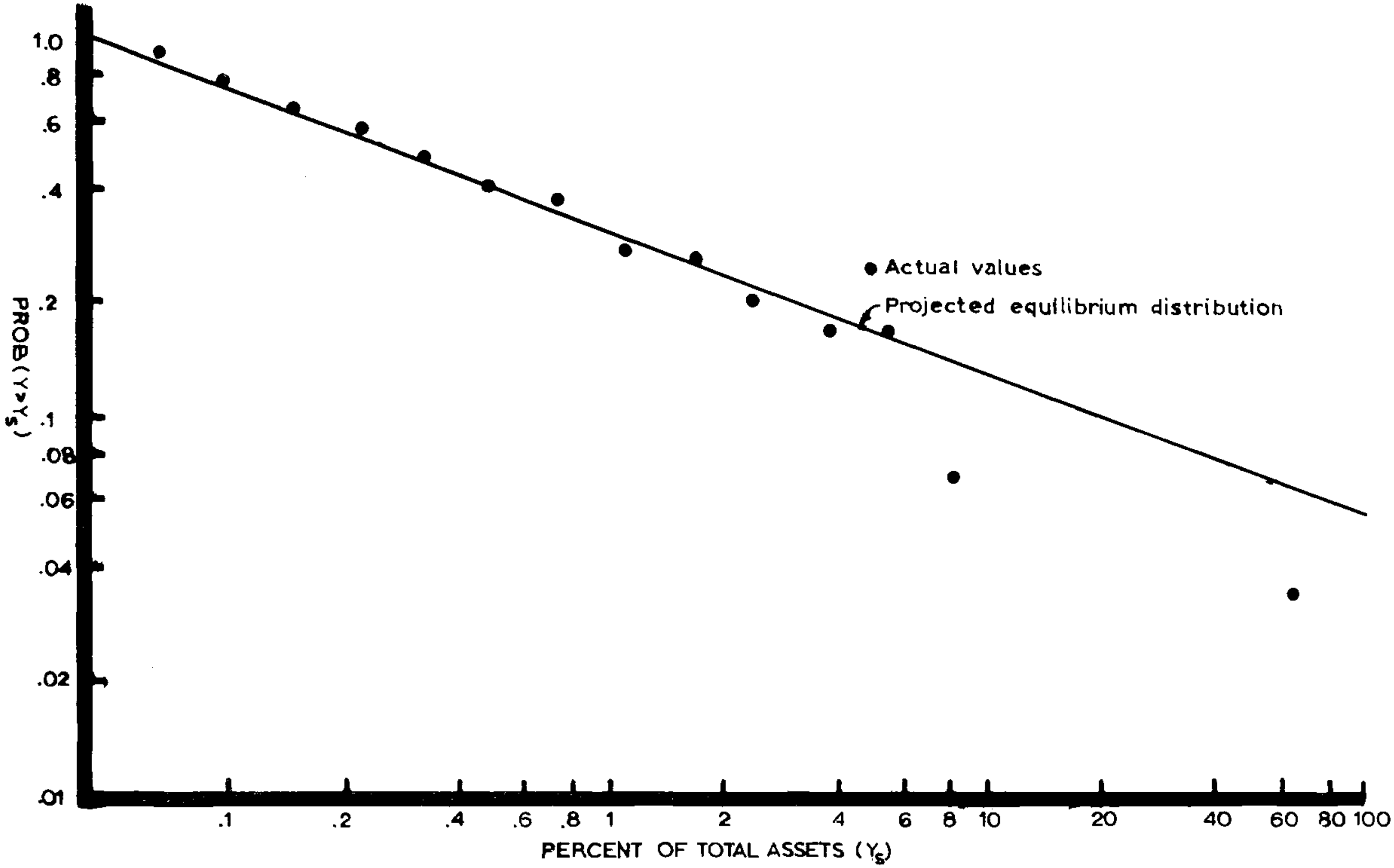
where the random variable y_s represents percentages of the total assets of the firms under consideration in each period of time.

In Figures 8 and 9 we can see in a log-log scale how equation (4-2) fits the actual size distribution of firms in the computing accounting industry in 1965 and 1966. The chi-square test for goodness of fit revealed no significance at 25 percent level of confidence for both curves.



Projected Equilibrium Distribution Function. Computing Accounting Industry, 1965. Actual Size Distribution For 1965 Plotted For Comparation.

FIG 8



Projected Equilibrium Distribution Function Computing Accounting Industry, 1966. Actual Size Distribution For 1966 Is Plotted For Comparition.

FIG. 9

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In this paper we have described a stochastic process for the derivation of the size distribution of firms in a given industry. In this process our basic postulate was that the probabilities of transition of firms through an enumerable infinity of size class ranges above some size limit were independent of the size of the firms themselves. As discussed above, the structure of leading firms within an industry will generally tend towards a Pareto type distribution function.

As a test of the model, we examined the computing accounting industry in the United States. We found that the behavior of the model is consistent with the empirical observations that one might expect: a tendency towards concentration and at the same time a tendency by part of the leading firms within the industry to reach a stable distribution as far as size of the firms is concerned.

It would appear that the application of this model does not lead to wrong results, and therefore increases the plausibility of the use of stochastic processes in the study of industrial structures.

Recommendations

Recommendations relating to this study and its implications for further research in the general area of the analyzing growth and size distribution of firms are several:

1. Future studies in this area could be concerned with proving the validity of our models in other industries.

2. It should be ascertained for which industries our assumptions would not give reliable results, the causes of this failure, and an economic interpretation of such failure.

3. Stochastic models of growth of firms that embody as much knowledge as we have about the underlying processes should be developed.

4. The application of this model to other areas might give reliable results if the above assumptions are considered.

5. The possible implication which the use of stochastic models could have on actual public policy which is based on static equilibrium analysis should be studied.

APPENDIX

ASSET SIZES OF THE COMPUTING ACCOUNTING INDUSTRY
1963-1966

	Million Dollars		
	1966	1965	1963
Astrodata, Inc.	25.1	15.7	4.9
Systron Doumer Corp.	9.7	7.4	5.5
Data Products Corp.	6.1	4.9	2.08
Coleman Eng. Co.	3.3	3.9	3.9
Whittaker Corp.	42.7	22.9	18.9
Scientific Data Systems	65.8	42.9	5.6
General Gilbert Co.	3.3	4.6	2.09
Soroban Eng., Inc.	5.8	4.9	4.8
Milgo Electronics Corp.	5.1	5.1	3.1
Victor Comptometer	83.5	71.4	62.4
Scan Instrument Corp.	5.4	5.1	4.4
Anelex Corp.	14.2	13.7	10.1
Electronic Corp. of America	9.2	8.5	7.8
Bendix Corp.	662.0	485.5	436.5
Burroughs Corp.	509.1	443.5	390.2
R. C. Allen Business Machine	--	5.6	5.2
Control Data Corp.	273.7	208.4	71.3
Honeywell, Inc.	772.0	598.1	468.0
Ultronic Systems Corp.	23.4	17.0	5.4
Digitronic Corp.	7.2	7.1	8.8
I.B.M.	4660.8	3745.0	1984.5
A.V.M. Corp.	26.8	26.2	20.0
Bunker Ramo Corp.	63.5	55.4	52.0
General Battery Ceramic	33.3	36.1	26.7
Olivetti Underwood Corp.	--	249.7	261.8
S.C.M. Corp.	146.3	96.9	88.4
Universal Controls, Inc.	38.9	35.6	34.3
Potter Instruments	10.5	8.6	7.8
National Cash Register Co.	566.4	494.1	462.5
Rockwell Mfg. Co.	149.6	137.6	124.0
Keltec Industries, Inc.	6.5	7.7	4.7
Digital Equipment Corp.	15.1	--	--
Electronics Memories Co.	5.3	--	--
Total Assets:	8249.6	6869.1	4588.4

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